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# PLANE ANALOG OF SPONTANEOUS SWIRLING

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*A plane analog of the problem of spontaneous swirling — the occurrence of a free transverse flow due to disturbance of the initial plane-parallel flow — is considered. It is shown that in flows with circular streamlines between coaxial cylinders, loss of stability can result in the occurrence of axial flow that is axisymmetric on the average (averaging over the axial coordinate and the azimuthal angle) because of the countergradient transfer of the axial momentum component by Reynolds stresses.*

The problem of spontaneous swirling was considered in [1–3], and it is formulated as follows: can rotational symmetric flow arise in the absence of obvious external sources of rotation, i.e., under conditions in which axisymmetric flow without rotation is a priori possible? The occurrence of free cross flow due to disturbance of the initial plane–parallel flow can be considered a plane analog of this phenomenon. A more exact formulation of this problem is given in [2, 3]. The formulation proposed there ensures strict control of the kinematic flow of the axial momentum component, which eliminates inflow of swirling liquid in the region considered. In the case of a plane analog, this formulation ensures strict control of the kinematic flow of the transverse momentum component and eliminates liquid inflow with transverse momentum through the lateral surface. The occurrence of transverse flow is considered as bifurcation of the initial plane–parallel flow as a result of loss of stability [1].

In [2, 3], it is shown that the bifurcation "axisymmetric flow–rotational symmetric flow" (and the corresponding plane analog of such transition [3]) for disturbances of the same symmetry as the initial flow does not take place for an arbitrary compressible liquid with varying viscosity.

In the present paper, we give an example of occurrence of transverse (axial) flow with loss of stability of the initial plane–parallel flow of an inviscid incompressible liquid with circular streamlines between two coaxial cylinders.

In the generally accepted notation (liquid density $\rho = 1$), the equations for such flows in cylindrical coordinates $r$, $\varphi$, $z$ have the form

$$Du - v^2/r = -p_r, \qquad Dv + uv/r = -p_\varphi/r; \tag{1}$$

$$Dw = -p_z, \qquad ru_r + u + v_\varphi + rw_z = 0, \tag{2}$$

where $D = \partial/\partial t + u\partial/\partial r + (v/r)\partial/\partial\varphi + w\partial/\partial z$; the subscripts indicate derivatives with respect to the corresponding variables; the velocity components $\mathbf{v} = (u, v, w)$.

We consider the disturbance of the initial flow $v_0 = v_0(r)$, $u_0 = 0$, $w_0 = 0$, $p_0 = p_0(r)$, and ${p_0}_r = {v_0}^2/r$, where $v_0(r)$ is an arbitrary function of $r_1 \leqslant r \leqslant r_2$ confined between circular concentric cylinders. Infinitesimal disturbances satisfy linearized equations (1) and (2):

$$D_0 u - 2v_0 v/r = -p_r, \qquad D_0 v + \Omega u = -p_\varphi/r; \tag{3}$$

$$D_0 w = -p_z, \qquad ru_r + u + v_\varphi + rw_z = 0; \tag{4}$$

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$$D_0 = \frac{\partial}{\partial t} + (v_0/r)\frac{\partial}{\partial \varphi}, \qquad \Omega = v_{0r} + v_0/r. \tag{5}$$

Solutions of system (3)–(5) must satisfy the boundary conditions $u = 0$ at $r = r_1$ and $r = r_2$.

Many papers have been devoted to studying this problem, and the behavior of solutions with variation in the initial velocity field $v_0(r)$ has been studied in detail. After disturbances are found by solution of the formulated problem, it is possible to compute the Reynolds stresses $z$ and $\varphi$ averaged over $\sigma_{\varphi r} = -\langle vu \rangle$ and $\sigma_{zr} = -\langle wu \rangle$ and to determine secondary (of the second order of smallness) flow $\langle u_2 \rangle = u_2(r,t)$, $\langle v_2 \rangle = v_2(r,t)$, and $\langle w_2 \rangle = w_2(r,t)$, averaged over the same variables. For these quantities from (1) and (2) in a second approximation we obtain

$$u_2(r,t) = 0, \qquad v_{2t} = -(r^2\langle vu \rangle)_r/r^2, \qquad w_{2t} = -(r\langle wu \rangle)_r/r. \tag{6}$$

In studies of the stability of the flows considered, one usually calculates the value of $\langle vu \rangle$ because, with allowance for viscosity, it governs the torque. The value of $\langle wu \rangle$ appears to have never been calculated. This is due to the fact that in corresponding experiments, the length of the cylinders is always limited, and the question of occurrence of flow that is axial and transverse to the initial flow does not arise under such conditions. However, if this axial flow (in an indefinitely long tube) is treated as an analog of spontaneous swirling, it makes sense to determine it. We note that the assumption that such flow can arise is put forward in [1].

To determine the quantity of interest, we use the well-known results of studies of the stability of the flow considered (see, for example, [4]). We seek a solution of system (3)–(5) in the form of normal modes

$$(u_1, v_1, w_1, p_1) = \mathrm{Re}(U, V, W, P)\, e^{i(kz + m\varphi - \omega t)},$$

where the functions $U$, $V$, $W$, and $P$ depend only on $r$. To determine these quantities, from (3)–(5) we obtain the following system of ordinary differential equations:

$$i\lambda U - 2v_0 V/r = -P_r, \qquad i\lambda V + \Omega U = -imP/r; \tag{7}$$

$$i\lambda W = -ikP, \qquad rU_r + U + imV + ikrW = 0 \tag{8}$$

with the boundary conditions $U(r_1) = U(r_2) = 0$. Here $\lambda = (mv_0/r) - \omega$. For given $k$ and $m$, system (7) and (8) together with the boundary conditions defines the eigenfunctions $U$, $V$, $W$, and $P$ and the eigenfrequencies $\omega$. If the frequency with $\mathrm{Im}\,\omega > 0$ is present in the spectrum, the initial flow is unstable.

Eliminating $V$, $W$, and $P$ from (7) and (8), for $U$ we obtain the equation

$$U'' + \frac{2-\alpha}{r}U' - \Big(K + \frac{\alpha}{r^2}\Big)U + \frac{A}{\lambda}U + \frac{B}{\lambda^2}U = 0. \tag{9}$$

Here $\alpha = (k^2r^2 - m^2)/(k^2r^2 + m^2)$, $A = 2k^2m\Omega/(k^2r^2 + m^2) - m\Omega'/r$, $B = 2k^2\Omega v_0/r$, and $K = k^2 + m^2/r^2$. If $U$ is found, $V$ and $W$ are given by equalities

$$V = \frac{i}{r^2K}\Big(m(rU)' + \frac{k^2r^2\Omega}{\lambda}U\Big); \tag{10}$$

$$W = \frac{ik}{rK}\Big((rU)' - \frac{m\Omega}{\lambda}U\Big). \tag{11}$$

To calculate Reynolds stresses, it is convenient to bring Eq. (9) to a form that does not contain the first derivative. We set

$$U = (K/r)^{1/2}Y.$$

Then, from (9) we obtain

$$Y'' + \Big(g(r) - K + \frac{A}{\lambda} + \frac{B}{\lambda^2}\Big)Y = 0, \tag{12}$$

where $g(r) = [(2 - 3\alpha)/2 + 2m^2k^2r^2/(m^2 + k^2r^2)^2]/r^2$.

The Reynolds stresses are defined as

$$\sigma_{\varphi r} = -\langle v_1 u_1 \rangle = -(1/2)\,\mathrm{Re}\,(VU^*)\,\exp(2qt),$$
$$\sigma_{zr} = -\langle w_1 u_1 \rangle = -(1/2)\,\mathrm{Re}\,(WU^*)\,\exp(2qt).$$

Hence, taking into account (9)–(12), we obtain formulas that define these quantities ($q = \mathrm{Im}\,\omega$):

$$(r^2\sigma_{\varphi r})_r = -\frac{q}{2}\exp(2qt)\Big[m|Y|^2(A/|\lambda|^2 + 2B\,\mathrm{Re}\,\lambda/|\lambda|^4) - k^2\frac{d}{dr}(r\Omega|Y|^2/|\lambda|^2)\Big]; \qquad (13)$$

$$(r\sigma_{zr})_r = -\frac{q}{2}\exp(2qt)\Big[k|Y|^2(A/|\lambda|^2 + 2B\,\mathrm{Re}\,\lambda/|\lambda|^4) + km\frac{d}{dr}(\Omega|Y|^2/r|\lambda|^2)\Big]. \qquad (14)$$

Formula (13) is given in [4]. For the problem considered, the Reynolds stress defined by formula (14) is important, because it leads to the occurrence of axial flow in a second approximation according to Eqs. (6). We note that although axisymmetric disturbances are known to grow most rapidly, they do not generate stresses required for the occurrence of axial flow because for $m = 0$, we have $A = 0$ and $q = 0$, or $\mathrm{Re}\,\lambda = 0$. This follows from Eq. (12). If $k = 0$, axial flow also does not occur, and this agrees with [3]. To show the possibility of transverse flow occurring, it suffices to consider initial data that correspond to any growing mode with nonzero $k$ and nonzero $m$. Then, $w_2$ is determined by integrating the third equation of (6) with respect to time with the initial condition $w_2(0, r) = 0$. From these equations it follows that there are no full expenditure and axial momentum components. A layered flow (at least, two layers) with opposite velocities increasing with time arises.

Hence it follows that the given flow involves countergradient transfer of the axial momentum component, which is sometimes associated with the occurrence of "negative" viscosity. This implies the following: if one tries to represent Reynolds stress using the "turbulent" viscosity coefficient and the strain tensor of mean flow, then, in the given flow, at least in a neighborhood of the boundary between the layers, the "turbulent" viscosity coefficient turns out to be negative. Apparently, this description is not adequate. Nevertheless, this terminology is widely used. These phenomena are known to arise in flows of planetary scale and in some rather complex flows (for example, the Ranque effect — the separation of hot and cold flows in a rotating liquid). The occurrence of this phenomenon in a simple flow such as the one considered here seems surprising and deserves further investigation. For a comprehensive study of the secondary flow resulting from instability and the stability of this mode, it is necessary to take into account nonlinearity and viscosity.

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## REFERENCES


1. M. A. Gol'dshtik, E. M. Zhdanova, and V. N. Shtern, "Spontaneous swirling of a submerged jet," *Dokl. Akad. Nauk SSSR*, **277**, No. 4, 815–818 (1984).
2. B. A. Lugovtsov, "Is spontaneous swirling of axisymmetric flow possible?" *Prikl. Mekh. Tekh. Fiz.*, **35**, No. 2, 50–54 (1994).
3. Yu. G. Gubarev and B. A. Lugovtsov, "Spontaneous swirling in axisymmetric flows," *Prikl. Mekh. Tekh. Fiz.*, **36**, No. 4, 52–59 (1995).
4. V. A. Vladimirov, "Analogy of the effects of stratification and rotation," in: *Nonlinear Problems of the Theory of Surface and Internal Waves* [in Russian], Nauka, Novosibirsk (1985), pp. 270–312.